Int J Theor Phys (2010) 49: 25–30
DOI 10.1007/s10773-009-0174-z

On Kantowski-Sachs Cosmic Strings Coupled with Maxwell Fields in Bimetric Relativity

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Received: 11 July 2009 / Accepted: 14 October 2009 / Published online: 23 October 2009

Abstract Kantowski-Sachs model is studied with source cosmic cloud strings coupled with electromagnetic field in Rosen's (Gen. Relativ. Gravit. 4:435, 1973) bimetric theory of relativity. It is shown that there is no contribution from Maxwell fields in this theory. Hence geometric string and vacuum cosmological models are established.

Keywords Cosmic strings · Bimetric theory · Maxwell fields

1 Introduction

Rosen [14] proposed the bimetric theory of relativity to remove some of the unsatisfactory features of the general theory of relativity, in which there exist two metric tensors at each point of space time, viz., a Riemannian metric tensor g_{ij}, which describes gravitation and the background metric γ_{ij}, which enters into the field equations and interacts with g_{ij} but does not interact directly with matter. One can regard γ_{ij} as describing the geometry that exists if matter were not present. Accordingly, at each space time point one has two line elements

$$ds^2 = g_{ij} dx^i dx^j \tag{1}$$

and

$$d\sigma^2 = \gamma_{ij} dx^i dx^j \tag{2}$$

where ds is the interval between two neighbouring events as measured by a clock and a measuring rod. The interval $d\sigma$ is an abstract or a geometrical quantity which is not directly measurable. One can regard it as describing the geometry that exists if no matter were present. Moreover, the bimetric theory also satisfied the covariance and equivalence principles: the formation of general relativity. The theory agrees with the present observational

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facts pertaining to general relativity [7]. As in general relativity, the variational principle also leads to the conservation law

$$T^{ij}_{;j} = 0 \tag{3}$$

where (;) denotes covariant differentiation with respect to g_{ij}. Accordingly the geodesic equation of a test particle is the same as that of general relativity.

The field equations of bimetric theory of gravitation proposed by Rosen [14] are

$$N^i_j - \frac{1}{2}N\delta^i_j = -8\pi\kappa T^i_j \tag{4}$$

where

$$N^i_j = \frac{1}{2}\gamma^{ab}\left(g^{hi}g_{hj1a}\right)_{1b}$$

and

$$N = N^i_i = N^1_1 + N^2_2 + N^3_3 + N^4_4,$$

$$g = \det(g_{ij}), \qquad \gamma = \det(\gamma_{ij}), \qquad \kappa = \left(\frac{g}{\gamma}\right)^{1/2}$$

A vertical bar (1) denotes the covariant differentiation with respect to γ_{ij} and T^i_j is the energy momentum tensor of the matter.

Rosen [14–16], Yilmaz [20], Karade and Dhoble [6], Karade [7], Israelit [2–4] are some of the eminent authors who have studied several aspects of bimetric theory of gravitation. In particular Mohanty and Sahoo [10] and Mohanty et al. [11] have established the non-existence of anisotropic spatially homogeneous Bianchi type cosmological models in bimetric theory when the source of gravitation is governed by either perfect fluid or mesonic perfect fluid. Reddy [12] have discussed the non-existence of anisotropic spatially homogeneous Bianchi type-I cosmological model in bimetric theory of gravitation in case of cosmic strings and Reddy and Venkateswarlu [13] have shown the non-existence of anisotropic Bianchi type-I perfect fluid models in Rosen's bimetric theory. Sahoo [18] have studied spherically symmetric string cosmological models in bimetric theory.

Recently, there has been a lot of interest in cosmological model on the basis of Rosen's bimetric theory of gravitation. The purpose of Rosen's bimetric theory is to get rid of the singularities that occur in general relativity. In bimetric theory, according to Rosen [14], the background metric tensor γ_{ij} should not be taken as describing an empty universe but it should rather be chosen on the basis of cosmological consideration. Hence Rosen proposed that the metric γ_{ij} be taken as the metric tensor of a universe in which perfect cosmological principle holds. In accordance with this principle, the large scale structure of the universe presents the same aspect from everywhere in space and at all times. The fact, however, is that while taking the matter actually present in the universe, this principle is not valid on small scale structure due to irregularities in the matter distribution and also not valid on large scale structure due to the evolution of the matter. Therefore, we adopt the perfect cosmological principle as the guiding principle. It does not apply to g_{ij} and the matter in the universe but to the metric γ_{ij}. Hence γ_{ij} describes a space time of constant curvature.

It is interesting to note that magnetic field plays a significant role in cosmological models. Melvin [9] suggested in the cosmological solution for dust and electromagnetic field that during the evolution of the universe, the matter was in highly ionized state and smoothly coupled with magnetic field and consequently forms a neutral matter as a result of universe expansion.

The relativists use various symmetries to get physically viable information from the complicated structure of the field equations in Einstein's theory of relativity. The field equations of general relativity are nonlinear in ten unknowns (g_{ij}) and it is very difficult to obtain their exact solutions. The involvement of symmetry namely spherical or cylindrical or plane reduces the number of gravitational potentials g_{ij} and thus helps one in simplifying the field equations to some extent. A space-time that admits the three-parameter group of motions of Euclidian plane is said to possess plane symmetry and is called plane symmetric space-time. The origin of structure in the universe is one of the greatest mysteries even today. The present day observations indicate that the universe at large scale is homogeneous and isotropic and it is witnessing an accelerating phase as reported recently [1]. It is well known that an exact solution of general theory of relativity for homogeneous space times belongs to either Bianchi types or Kantowski-Sachs [17].

In this work Kantowski-Sachs plane symmetric cosmological model is studied in the context of bimetric relativity with cosmic strings coupled to an electromagnetic field and it is observed that electromagnetic field does not contribute to the energy momentum tensor in this theory. Hence singularity free geometric string and vacuum models are obtained.

2 Kantowski-Sachs Model and Field Equations

Consider the Kantowski-Sachs [5] metric in the form

$$ds^2 = dt^2 - A^2 dr^2 - B^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{5}$$

where the metric potentials A and B are functions of cosmic time t only.

The background metric of flat space time is

$$d\sigma^2 = dt^2 - dr^2 - \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{6}$$

The energy momentum tensor for a cosmic cloud strings as Letelier [8] and Stachel [19] coupled with an electromagnetic field is written as

$$T^i_j = T^i_{j\text{strings}} + E^i_{j\text{mag}} \tag{7}$$

where

$$T^i_{j\text{strings}} = \rho U^i U_j - \lambda X^i X_j \tag{8}$$

together with $U^i U_i = 1 = -X^i X_i$ and $U^i X_i = 0$ and

$$E^i_{j\text{mag}} = -F_{jr}F^{ir} + \frac{1}{4}F_{ab}F^{ab}g^i_j \tag{9}$$

where $E^i_{j\text{mag}}$ is electromagnetic energy tensor, F^i_j the electromagnetic field tensor, U^i is the four velocity of the string cloud, X^i represents a direction of anisotropy, i.e., the direction of strings and ρ and λ are the rest energy density of the cloud of strings and the tension density of the string cloud respectively.

In the co-moving coordinate system taking the string in radial direction (8) takes the form

$$T^1_{1\text{strings}} = \lambda, \qquad T^4_{4\text{strings}} = \rho$$

and

$$T^i_{j\text{strings}} = 0 \quad \text{for } i, j = 2, 3 \text{ and for } i \neq j$$

The electromagnetic field is considered to be along the radial direction so that the only non-vanishing components of electromagnetic field tensor F_{ij} is F_{23}.

The first set of Maxwell's equation

$$F_{[ij,k]} = F_{ij,k} + F_{jk,i} + F_{ki,j} = 0 \tag{10}$$

leads to the result $F_{23} = \text{constant} = H(\text{say})$. Then

$$E^1_1 = -E^2_2 = -E^3_3 = E^4_4 = \frac{H^2}{2B^4} = \eta(\text{say}) \tag{11}$$

The Rosen's field equations (4) for the metric (5) and (6) with the help of (7) becomes

$$\left(\frac{A_4}{A}\right)_4 - 2\left(\frac{B_4}{B}\right)_4 = 16\pi\kappa(\lambda + \eta) \tag{12}$$

$$\left(\frac{A_4}{A}\right)_4 = 16\pi\kappa\eta \tag{13}$$

$$\left(\frac{A_4}{A}\right)_4 + 2\left(\frac{B_4}{B}\right)_4 = -16\pi\kappa(\rho + \eta) \tag{14}$$

where the suffix 4 following an unknown function denotes ordinary differentiation with respect to t.

Equations (12)–(14) gives

$$8\pi\kappa(\rho - \lambda + 2\eta) = 0 \tag{15}$$

2.1 Vacuum Model

If $\rho = 0 = \lambda$ equations (12) to (15) implies

$$\left(\frac{A_4}{A}\right)_4 = \left(\frac{B_4}{B}\right)_4 = 0 \quad \text{and} \quad \eta = 0 \tag{16}$$

which shows that there is no contribution from Maxwell fields to Kantowski-Sachs plane symmetric model in bimetric relativity.

Hence Kantowski-Sachs vacuum model in bimetric relativity can be expressed, after a proper choice of coordinates and constants, in the form

$$ds^2 = dt^2 - e^{2^{a_1 t}} dr^2 - e^{2^{a_2 t}} \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{17}$$

where a_1 and a_2 are constants of integration. It may be observed that for $a_1 = a_2 = a$

$$ds^2 = dt^2 - e^{2^{at}} \left(dr^2 + d\theta^2 + \sin^2\theta d\phi^2\right) \tag{18}$$

This vacuum model represents Robertson-Walker flat model, which expands uniformly along space directions with time. The rate of expansion depends on the signature of the parameter a.

2.2 Geometric String Model ($\rho = \lambda$)

In case of geometric string ($\rho = \lambda$) (15) gives

$$\eta = 0 \tag{19}$$

Using this equation (13) reduces to

$$\left(\frac{A_4}{A}\right)_4 = 0 \tag{20}$$

Equation (20) gives

$$A = \exp(a_1 t + a_2) \tag{21}$$

By the help of (19) and (20) the field equations (12) to (14) for geometric string reduces to

$$\left(\frac{B_4}{B}\right)_4 = -8\pi\kappa\rho \tag{22}$$

So, finally there is a single equation involving two unknowns, hence to get a physically meaningful solution let

$$B = (a_3 t + a_4)^n \exp(a_3 t + a_4) \tag{23}$$

and

$$8\pi\kappa\rho = 8\pi\kappa\lambda = \frac{n a_3^2}{(a_3 t + a_4)^2} \tag{24}$$

where 'n' is an arbitrary constant and a_i's are constants of integration.

The corresponding cosmological model can now be written in the form

$$ds^2 = dt^2 - \exp\left[2(a_1 t + a_2)\right]dr^2 - \left\{(a_3 t + a_4)^{2n} \exp\left[2(a_3 t + a_4)\right]\right\}\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{25}$$

Thus equations (25) together with (24) constitutes Kantowski-Sachs geometric string model in bimetric theory of gravitation.

If $n = 0$ the above model reduces to vacuum cosmological model given by (18).

3 Conclusions

Kantowski-Sachs models are considered as possible candidates for an early era in cosmology. Here it is shown that for Kantowski-Sachs model Maxwell fields do not exist in Rosen's [14] bimetric theory of gravitation. Hence a vacuum model is constructed which is free from singularity and reduces to a flat space-time when $t = 0$ and the geometric string model is established. It is observed that if 'n' is negative the model (25) has a singularity. Since bimetric theory has no initial singularity the arbitrary constant 'n' should be non-negative.

Acknowledgement The author would like to convey his sincere thanks and gratitude to the anonymous referee for his useful and kind suggestions for improvement of the paper.

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